

Non-standard eigenvalue problems for perturbed p -Laplacians

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Abstract

This paper is devoted to multi-parameter eigenvalue problems for perturbed p -Laplacians, modelling travelling waves for a class of nonlinear evolution PDE. Dispersion relations between the eigen-parameters, the existence of eigenvectors and positive eigenvectors, variational principles for eigenvalues of perturbed p -Laplacians and constructing analytical solutions are the main subject of this paper. Besides the p -Laplacian-like eigenvalue problems we also deal with new and non-standard eigenvalue problems, which can not be solved by the methods used in nonlinear eigenvalue problems for p -Laplacians and similar operators. We do both: extend and use classical variational and analytical techniques to solve standard eigenvalue problems and suggest new variational and analytical methods to solve the non-standard eigenvalue problems we encounter in the search for travelling waves.

Keywords: Travelling waves, perturbed p -Laplacians, eigenvalues, eigenfunctions, variational principles, critical points.

AMS subject Classifications: Primary 49R50, 47A75, 35p15; Secondary 34L15, 35D05, 35J60

1 Introduction

In this paper we study a class of non-standard eigenvalue problems, which naturally arise when we search travelling waves for evolution p -Laplacian

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equations. The point of departure for the problems studied in this paper is the evolution (p, q) -Laplacian equation in the following form:

$$iv_t - \operatorname{div}(|\nabla v|^{p-2} \nabla v) = \lambda |v|^{q-2} v, v|_{\partial Q} = 0,$$

where $v := v(t, x, y)$, $t > 0$, λ is a parameter and $Q = (0, 1) \times \mathbb{R}$ is an infinite rectangle in \mathbb{R}^2 . We can think this equation as a generalized non-linear Schrödinger equation because it subsumes the evolution non-linear Schrödinger equation in the particular case $p = 2$.

It should be noted that studying travelling waves reduces this problem to a multi-parameter eigenvalue problem and the most difficult eigenvalue problems arise, when $p = q$ (see J. P. G. Azorero, I. P. Alonso [4]). For this reason, in this paper we restrict ourselves to the case $\mathbf{p} = \mathbf{q}$ and start with the following problem (see [8]):

$$\begin{aligned} iv_t - \operatorname{div}(|\nabla v|^{p-2} \nabla v) &= \lambda |v|^{p-2} v, \quad p > 1 \\ v|_{\partial Q} &= 0. \end{aligned} \tag{1.1}$$

Now, we look for wave solutions to equation (1.1) in the form $v(t, x, y) = e^{i(wt - ky)} u(x)$, where $x \in (0, 1)$, $y \in \mathbb{R}$, $u(x)$ is a real-valued function and $(w, k) \in \mathbb{R} \times \mathbb{R}$. We have $\nabla v = (v_x, v_y) = e^{i(wt - ky)}(u', -iku)$ and $|\nabla v| = (k^2 u^2 + u'^2)^{1/2}$. For convenience we introduce the notation: $\nabla_k u := (ku, u')$. Then, $|\nabla v| = |\nabla_k u| = (k^2 u^2 + u'^2)^{1/2}$. By using this notation and putting $v(t, x, y) = e^{i(wt - ky)} u(x)$ into (1.1) we obtain that $v(t, x, y)$ is a solution to equation (1.1) if and only if u is an eigenvector of the following multi-parameter eigenvalue problem:

$$\begin{aligned} -wu + k^2 |\nabla_k u|^{p-2} u - (|\nabla_k u|^{p-2} u')' &= \lambda |u|^{p-2} u, \quad p > 1 \\ u(0) &= u(1) = 0. \end{aligned} \tag{1.2}$$

We note that in the case of $k = 0$ we obtain $|\nabla_0 u| = |u'|$ and (1.2) yields

$$\begin{aligned} -wu - (|u'|^{p-2} u')' &= \lambda |u|^{p-2} u, \\ u(0) &= u(1) = 0, \end{aligned}$$

which is a two-parameter eigenvalue problem for one-dimensional p -Laplacians. For this reason we refer to problem (1.2) as a multi-parameter eigenvalue problem for perturbed p -Laplacians. We now give the definition of solution to equation (1.2).

Definition 1.1 $0 \neq u \in W_0^{1,p}(0,1)$ is a solution to equation (1.2) if and only if

$$-w \int_0^1 uv \, dx + k^2 \int_0^1 |\nabla_k u|^{p-2} uv \, dx + \int_0^1 |\nabla_k u|^{p-2} u' v' \, dx = \lambda \int_0^1 |u|^{p-2} uv \, dx \quad (1.3)$$

holds for all $v \in W_0^{1,p}(0,1)$, where $W_0^{1,p}(0,1)$ is the Sobolev space (see [1] for Sobolev spaces). Here, the parameters (w, k, λ) are called eigen-parameters and the associated non-trivial function $u \in W_0^{1,p}(0,1)$ is called the eigenfunction. Besides the eigenfunctions, we are interested in positive eigenfunction, too. We note that eigenvalue problems (1.3) can be split into two groups with respect to w .

I) The case: $w \neq 0$,

II) The case $w = 0$.

In the case **II)** problem (1.3) can be written in the following form:

$$k^2 \int_0^1 |\nabla_k u|^{p-2} uv \, dx + \int_0^1 |\nabla_k u|^{p-2} u' v' \, dx = \lambda \int_0^1 |u|^{p-2} uv \, dx.$$

This problem actually contains two quite different type of eigenvalue problems:

a) Standard eigenvalue problems: In these problems we fix k and study λ , i.e., we are interested in the dispersion relation $\lambda(k)$.

b) Non-standard eigenvalue problems: Here we search k for a fixed λ .

At this point we must note that the case b) is the hardest problem we study in this paper. Moreover, this problem is not a standard eigenvalue problem as treated in the context of Nonlinear Analysis. This case is the main concern of this paper and we study it separately in the last two Sections.

First, we observe that searching for $0 \neq u \in W_0^{1,p}(0,1)$ satisfying (1.3) is equivalent to finding critical points of the functional

$$F(u) = -\frac{w}{2} \int_0^1 u^2 \, dx + \frac{1}{p} \int_0^1 |\nabla_k u|^p \, dx - \frac{\lambda}{p} \int_0^1 |u|^p \, dx.$$

In what follows, we denote $X := W_0^{1,p}(0,1)$, which is a Banach space. $F' : X \rightarrow X^*$ will denote the Fréchet derivative of F , where X^* is the dual of the space X . It is known that the existence of Fréchet derivative implies the existence of directional (Gateaux) derivative. By using the definition of

Gateaux derivative, we can obtain

$$\begin{aligned}
\langle F'(u), v \rangle &= \frac{d}{dt} F(u + tv) \Big|_{t=0} = \frac{d}{dt} \left[-\frac{w}{2} \int_0^1 (u + tv)^2 dx + \right. \\
&\quad \left. \frac{1}{p} \int_0^1 \left(k^2 (u + tv)^2 + (u' + tv')^2 \right)^{\frac{p}{2}} dx - \frac{\lambda}{p} \int_0^1 |(u + tv)^p dx \right] \Big|_{t=0} = \\
&\quad -w \int_0^1 uv dx + \frac{1}{p} \int_0^1 \frac{p}{2} \left(k^2 u^2 + u'^2 \right)^{\frac{p-2}{2}} \left(2k^2 uv + 2u'v' \right) dx - \\
&\quad \lambda \int_0^1 |u|^{p-2} uv dx = -w \int_0^1 uv dx + k^2 \int_0^1 |\nabla_k u|^{p-2} uv dx + \\
&\quad \int_0^1 |\nabla_k u|^{p-2} u'v' dx - \lambda \int_\Omega |u|^{p-2} uv dx.
\end{aligned}$$

Hence, $u \in X$ is a solution to (1.3) if and only if u is a free critical point for $F(u)$, i.e., $\langle F'(u), v \rangle = 0$, for all $v \in X$, where $\langle F'(u), v \rangle$ denotes the value of the functional $F'(u)$ at $v \in X$. Moreover, by Sobolev's embedding theorem $W_0^{1,p}(0, 1)$ is compactly embedded in $C[0, 1]$ and consequently, $W_0^{1,p}(0, 1)$ is compactly embedded in $L_q(0, 1)$ for all $q \in [1, \infty)$ (see [1]). Therefore, the functional $F(u)$ is well defined for all $u \in W_0^{1,p}(0, 1)$.

The rest of this paper will be organized as follows. In Section 2 we study the $\lambda(k)$ dependence by using the Ljusternik-Schnirelman critical point theory (Theorem 2.1). We also prove a theorem (Theorem 2.3) about the existence of positive eigenfunctions and the localization of eigen-parameters (w, k, λ) . Section 3 is mainly devoted to constructing the analytical solutions and the dispersion relations $k(\lambda)$ based on the analytical solutions (Theorem 3.1). Here, we modify the existing methods and apply them to new problems. In the last section we discuss alternative variational methods for describing the dispersion relations $k(\lambda)$.

2 The structure of the eigen-parameters (w, k, λ) : General results

In this section we first study the problem

$$k^2 \int_0^1 |\nabla_k u|^{p-2} uv dx + \int_0^1 |\nabla_k u|^{p-2} u'v' dx = \lambda \int_0^1 |u|^{p-2} uv dx, \quad (2.1)$$

and the dispersion relation $\lambda(k)$ (the case **II-a**). We also present our main results about the structure of the eigen-parameters (w, k, λ) and associated eigenfunctions.

We have already noticed that for a fixed k problem (2.1) is a new class of standard eigenvalue problem for perturbed p -Laplacians. In the case of $k = 0$ problem (2.1) coincides with the typical one-dimensional eigenvalue problem for p -Laplacians:

$$\begin{aligned} -(|u'|^{p-2}u')' &= \lambda|u|^{p-2}u, \quad p > 1 \\ u(0) &= u(1) = 0. \end{aligned} \tag{2.2}$$

The eigenvalue problems for p -Laplacians have been studied by many authors (see [2], [3], [4], [9], [14] and references therein). However, when $k \neq 0$ then we deal with (2.1) which is different from the typical one-dimensional p -Laplacian eigenvalue problem (2.2). Our first observation is given in Theorem 2.1. Since many facts in this theorem are proved in the same way as those in the classical p -Laplacian eigenvalue problems we present only a sketch of the proof.

Theorem 2.1 *Let us fix $k \in \mathbb{R}$. Then,*

a) there exists an infinite sequence of eigenvalues $\lambda_n(k)$ for problem (2.1), arranged as

$$0 < \lambda_1(k) < \lambda_2(k) \leq \dots \leq \lambda_n(k) \leq \dots \text{ and } \lambda_n(k) \rightarrow +\infty \text{ as } n \rightarrow \infty,$$

where $\lambda_1(k) < \lambda_2(k)$ follows from c).

b) $0 < \lambda_1(\Delta_p) \leq \lambda_1(k)$ and $|k|^p \leq \lambda_1(k)$,

where $\lambda_1(\Delta_p)$ denotes the first eigenvalue of problem (2.2).

c) $\lambda_1(k)$ is simple, isolated and $\lambda_2(k)$ is the second eigenvalue of problem (2.1)

A sketch of the Proof. a) The proof is based on the Ljusternik-Schnirelman critical point theory (see [19], Chapter 44). Let us define $G_k(u) := \frac{1}{p} \int_0^1 |\nabla_k u|^p dx$ and $\Phi(u) := \frac{1}{p} \int_0^1 |u|^p dx$. Consider the following eigenvalue problem:

$$\Phi'(u) = \mu G'_k(u), \quad u \in S_{G_k}, \quad \mu \in \mathbb{R}, \tag{2.3}$$

where $S_{G_k} = \{u \in X \mid G_k(u) = 1\}$. Clearly, equation (2.3) is the same as equation (2.1) with $\mu = \frac{1}{\lambda}$. It is well known that to find a sequence μ_n of the eigenvalues of problem (2.3), it is sufficient to check the following basic conditions (see [19], p. 325 and p. 328):

H1. Let X be a reflexive Banach space. F and G are the even functionals such that $F, G \in C^1(X, \mathbb{R})$ and $F(0) = G(0) = 0$.

H2. $F' : X \rightarrow X^*$ is strongly continuous (i.e. $u_n \rightharpoonup u$ implies $F'(u_n) \rightarrow F'(u)$) and $\langle F'(u), u \rangle = 0$, $u \in \overline{\text{co}}S_G$ implies $F(u) = 0$, where $\overline{\text{co}}S_G$ denotes the closure of the convex hull of the set S_G .

H3. $G' : X \rightarrow X^*$ is continuous, bounded and satisfies the following condition:

$$u_n \rightharpoonup u, \quad G'(u_n) \rightharpoonup v, \quad \langle G'(u_n), u_n \rangle \rightarrow \langle v, u \rangle$$

implies $u_n \rightarrow u$ as $n \rightarrow \infty$, where $u_n \rightharpoonup u$ denotes the weak convergence in X .

H4. The level set S_G is bounded and $u \neq 0$ implies,

$$\langle G'(u), u \rangle > 0, \quad \lim_{t \rightarrow +\infty} G(tu) = +\infty, \quad \inf_{u \in S_G} \langle G(u), u \rangle > 0.$$

Note that in our case, $F = \Phi$ and $G = G_k$. It has been shown in [9] that all conditions H1 – H4 are satisfied for the functionals $\Phi(u)$ and $G_0(u)$. If $k \neq 0$, then one can easily adapt the techniques of the proofs given in [9] to our case, to show that all of the above-given conditions are satisfied.

Now, we denote by $\mathcal{K}_n(k)$ the class of all compact, symmetric subsets K of G_k , such that $\text{gen } K \geq n$ (see [19], Chapter 44). Thus, for a fixed $k \in \mathbb{R}$, according to the Ljusternik-Schnirelman variational principle ([19], p. 326, Theorem 44.A) there exists a sequence of eigenvalues of problem (2.1), depending on k and arranged as:

$$0 < \lambda_1(k) \leq \lambda_2(k) \leq \dots \leq \lambda_n(k) \leq \dots$$

which are characterized by

$$\frac{1}{\lambda_n(k)} = \mu_n(k) = \sup_{K \subset \mathcal{K}_n(k)} \inf_{u \in K} \Phi(u).$$

Moreover, for all $k \in \mathbb{R}$

$$0 < \lambda_1(k) \text{ and } \lambda_n(k) \rightarrow +\infty \text{ as } n \rightarrow \infty.$$

b) It follows from

$$0 < \lambda_1(\Delta_p) = \inf_{0 \neq u \in W_0^{1,p}(0,1)} \frac{\int_0^1 |u'|^p dx}{\int_0^1 |u|^p dx} \leq \inf_{0 \neq u \in W_0^{1,p}(0,1)} \frac{\int_0^1 (k^2 u^2 + u'^2)^{\frac{p}{2}} dx}{\int_0^1 |u|^p dx} = \lambda_1(k)$$

and

$$|k|^p \int_0^1 |u|^p dx \leq \int_0^1 (k^2 u^2 + u'^2)^{\frac{p}{2}} dx$$

that $0 < \lambda_1(\Delta_p) \leq \lambda_1(k)$ and $|k|^p \leq \lambda_1(k)$.

c) The fact that $\lambda_1(k)$ is isolated is proved by the same method that is given in [14] to prove that the first eigenvalue of the p -Laplacian is isolated. Finally, to prove that $\lambda_2(k)$ is the second eigenvalue of problem (2.1) one

may follow the approach of the paper [3]. The simplicity result repeats the arguments from [2].

The case: I).

In this case $w \neq 0$ and our the main concern is problem (1.3) with $u > 0$:

$$k^2 \int_0^1 |\nabla_k u|^{p-2} uv \, dx + \int_0^1 |\nabla_k u|^{p-2} u' v' \, dx - \lambda \int_0^1 |u|^{p-2} uv \, dx = w \int_0^1 uv \, dx,$$

$$u > 0 \quad \text{in } (0, 1).$$

By using the scaling property, a solution to problem (1.3) can be obtained by a constrained minimization problem for the functional

$$E_{k,\lambda}(u) = \int_0^1 |\nabla_k u|^p \, dx - \lambda \int_0^1 |u|^p \, dx$$

on the Banach space $W_0^{1,p}(0, 1)$, restricted to the set

$$M = \left\{ u \in W_0^{1,p}(0, 1) \mid \int_0^1 u^2 \, dx = 1 \right\}.$$

The main idea is based on some regularity ideas, on the fact that $W_0^{1,p}(0, 1)$ is compactly embedded in $L_q(0, 1)$ for all $q \in [1, \infty)$ and on the following theorem.

Theorem 2.2 (see [17], Theorem 1.2) *Suppose X is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset X$ be a weakly closed subset of X . Suppose $E : M \rightarrow \mathbb{R} \cup +\infty$ is coercive on M with respect to X , that is*

*1) $E(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$, $u \in M$
and it is (sequentially) weakly lower semi-continuous on M with respect to X , that is*

2) for any $u \in M$, any sequence (u_n) in M such that $u_n \rightharpoonup u$ (weakly) in X there holds:

$$E(u) \leq \liminf_{n \rightarrow \infty} E(u_n).$$

Then E is bounded from below on M and attains its infimum in M .

Now we formulate and prove our main result in this section.

Theorem 2.3 *If either $(w, k, \lambda) \in \mathbb{R}_+ \times \mathbb{R} \times (-\infty, \lambda_1(k))$ or $(w, k, \lambda) \in \mathbb{R}_- \times \mathbb{R} \times (\lambda_1(k), +\infty)$, then problem (1.3) has a positive solution.*

Proof. Let us consider the condition $(w, k, \lambda) \in \mathbb{R}_+ \times \mathbb{R} \times (-\infty, \lambda_1(k))$. By this condition we have to prove that for a fixed $k \in \mathbb{R}$ and $\lambda < \lambda_1(k)$ problem (1.3) has a positive solution for any $w > 0$. We set in Theorem 2.2: $X = W_0^{1,p}(0, 1)$, $E(u) := E_{k,\lambda}(u)$ and $M = \{u \in W_0^{1,p}(0, 1) \mid \int_0^1 u^2 dx = 1\}$. Evidently, all conditions of Theorem 2.2 are satisfied. Particularly, by the Sobolev's embedding theorem M is a weakly closed set. Now, the existence of a non-trivial solution to problem (1.3) immediately follows from this theorem. The existence of a non-negative solution is obtained if we replace u by $|u|$. To prove that a non-negative solution is positive we use some regularity results for solutions to (1.3). We can do this in the three steps given below.

Step 1. We show that a solution to (1.3) belongs to $L^\infty(0, 1)$. To show this one can use the Moser iteration technique (see [11] or [9], pp. 1070-1073). Actually, one can repeat step by step the method which was applied to prove that the eigenfunctions for p -Laplacians are bounded (see [9]).

Step 2. Now, we prove that $u \in C^{1,\alpha}(0, 1)$ -Hölder continuously differentiable function with the exponent $0 \leq \alpha \leq 1$. The proof is based on the following fact: Let Ω be a bounded domain in \mathbb{R}^n and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function (see [9], p. 1074). Then if $g(x) := f(x, u(x)) \in L^\infty(0, 1)$ then a result of DiBenedetto [7] and Tolksdorf [18] states that a weak solution of the equation

$$-\Delta_p u(x) = f(x, u(x)) \text{ in } \Omega \quad (2.4)$$

is a $C^{1,\alpha}(\Omega)$ function.

Step 3. Finally, we use the following Harnack type inequality due to Trudinger ([12] and [9], p. 1075) to prove that $u > 0$.

Harnack inequality: Let $u \in W^{1,p}(\Omega)$ be a weak solution of (2.4) and for all $M < \infty$ and for all $(x, s) \in \Omega \times (-M, M)$ the condition

$$|f(x, s)| \leq b_1(x)|s|^{p-1} + b_2(x)$$

holds, where b_1, b_2 are nonnegative functions in $L^\infty(\Omega)$. Then if $0 \leq u(x) < M$ in a cube $K(3r) := K_{x_0}(3r) \subset \Omega$, there exists a constant C such that

$$\max_{K(r)} u(x) \leq C \min_{K(r)} u(x).$$

In our problem it is enough to check *Step 1*. Then the chain *Step 1* \Rightarrow *Step 2* \Rightarrow *Step 3* is obvious. We also note that the condition $\max_{K(r)} u(x) \leq C \min_{K(r)} u(x)$ means either: $u = 0$ or $u > 0$ in Ω . Since $\|u\|_{L_2(0,1)} = 1$ we obtain that $u > 0$ in Ω .

The rest of the paper is devoted to case **II-b**), i.e., we seek k for a fixed λ in problem (2.1).

3 Analytical solutions of two-parameter eigenvalue problems and dispersion relations

In this section we study the analytical solutions of the following problem:

$$\begin{aligned} k^2 |\nabla_k u|^{p-2} u - (|\nabla_k u|^{p-2} u')' &= \lambda |u|^{p-2} u, \quad p > 1 \\ u(0) &= u(1) = 0. \end{aligned} \quad (3.1)$$

It turns out that in some cases it is possible to find an analytical solution to problem (3.1). The construction of the analytical solutions also allows us to get some dispersion relations between k and λ . In the next Section, we will separately discuss the methods of describing the dispersion relations $k(\lambda)$ in the case when we can not find analytical solutions.

First we note that in the case $k = 0$ we deal with the following classical one-dimensional p -Laplacian eigenvalue problem which was fully studied by P. Drábek [10] (see also a paper of M. Del Pino, M. Elgueta, R. Manasevich [16]):

$$\begin{aligned} -(|u'|^{p-2} u')' &= \lambda |u|^{p-2} u, \quad p > 1 \\ u(0) &= u(1) = 0. \end{aligned} \quad (3.2)$$

All eigenfunctions and eigenvalues are given by $u_n(x) = c \lambda_n^{-\frac{1}{p}} \sin_p(\lambda_n^{\frac{1}{p}} x)$ and $\lambda_n = (n\pi_p)^p$, respectively. Here, $\pi_p := 2 \int_0^{(p-1)/p} \frac{ds}{(1 - \frac{s^p}{p-1})^{1/p}}$ and $\sin_p(x)$ is defined as an implicit function $\sin_p : [0, \frac{\pi_p}{2}] \rightarrow [0, (p-1)^{1/p}]$ by

$$\int_0^{\sin_p(x)} \frac{ds}{(1 - \frac{s^p}{p-1})^{1/p}} = x,$$

then it is extended by setting: $\widetilde{\sin}_p(x) := \sin_p(\pi_p - x)$, $x \in [\frac{\pi_p}{2}, \pi_p]$ and $\widetilde{\sin}_p(x) := -\sin_p(-x)$ for $x \in [-\pi_p, 0]$. Finally, $\sin_p : \mathbb{R} \rightarrow \mathbb{R}$ is defined as the $2\pi_p$ -periodic extension of $\widetilde{\sin}_p(x)$ to all of \mathbb{R} (see [10] and [16], and also the recent paper [5] for more interesting properties of $\sin_p(x)$).

To construct analytical solutions to problem (3.1) for some special cases, we also apply methods similar to those applied in the above mentioned papers. However, we have to modify some techniques of these papers which are not applicable to our problems. Next, we present a modified version of the methods, used in [10] and [16] to construct the analytical solutions to problem (3.2). Namely, this modified method will be applied to solve analytically problem (3.1) in some special cases.

Let us consider equation (3.2). For the sake of simplicity we assume that u and u' are positive. Thus we consider the following problem:

$$\begin{aligned} -((u')^{p-1})' &= \lambda u^{p-1}, \quad p > 1 \\ u(0) &= u(1) = 0. \end{aligned} \quad (3.3)$$

or equivalently

$$\begin{aligned} -(p-1)(u')^{p-2}u'' &= \lambda u^{p-1}, \\ u(0) &= u(1) = 0. \end{aligned}$$

By using the substitution $u' = v(u)$ we can reduce the order of this equation. Indeed, $u' = v(u)$ implies $u'' = vv'$. Then we have

$$-(p-1)v^{p-1}v' = \lambda u^{p-1},$$

$$v^{p-1}dv = -\frac{\lambda}{p-1}u^{p-1}du.$$

Integrating both sides we obtain

$$v^p = c - \frac{\lambda}{p-1}u^p \Leftrightarrow u'^p = c - \frac{\lambda}{p-1}u^p. \quad (3.4)$$

Now, by using the condition $u(0) = 0$ we can write $\int_0^{u(x)} \frac{dr}{\left(c - \frac{\lambda}{p-1}r^p\right)^{1/p}} = x$ and the substitution $s = c^{\frac{1}{p}}r$ yields

$$\int_0^{c^{-\frac{1}{p}}u(x)} \frac{ds}{\left(1 - \frac{\lambda}{p-1}s^p\right)^{1/p}} = x. \quad (3.5)$$

Note. At this point we have to note that, we can take λ out of the integrand by using the substitution $s = s_0\lambda^{-\frac{1}{p}}$ and then define the function $\sin_p(x)$. The authors in [10] and [16] follow this way. However, we will see below that, for problem (3.1) we also have a similar expression where we can not take λ out. That is why we need a modification of this method.

Now we apply a technique which will also be applied to solve more difficult equation (3.1). Let us define

$$F(\lambda, x) := \int_0^x \frac{ds}{\left(1 - \frac{\lambda}{p-1}s^p\right)^{1/p}}.$$

Evidently,

$$F(\lambda, \cdot) : \left[0, \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p}}\right] \rightarrow \left[0, \frac{\pi_p(\lambda)}{2}\right], \quad \text{where } \pi_p(\lambda) := 2 \int_0^{(\frac{p-1}{\lambda})^{1/p}} \frac{ds}{\left(1 - \frac{\lambda}{p-1}s^p\right)^{1/p}}.$$

We have $F(\lambda, 0) = 0$ and $F'_x(\lambda, x) = \frac{1}{(1 - \frac{\lambda}{p-1}x^p)^{1/p}} > 0$, $x \in [0, (\frac{p-1}{\lambda})^{1/p}]$. It means that there exists the inverse function $G(\lambda, \cdot) : [0, \frac{\pi_p(\lambda)}{2}] \rightarrow [0, (\frac{p-1}{\lambda})^{1/p}]$ defined by

$$\int_0^{G(\lambda, x)} \frac{ds}{(1 - \frac{\lambda}{p-1}s^p)^{1/p}} = x.$$

To extend the function $G(\lambda, x)$ we follow the same way, which has been applied to extend $\sin_p(x)$. Namely, $\tilde{G}(\lambda, x) := G(\lambda, \pi_p(\lambda) - x)$, $x \in [\frac{\pi_p(\lambda)}{2}, \pi_p(\lambda)]$ and $\tilde{G}(\lambda, x) = -\tilde{G}(\lambda, -x)$ for $t \in [-\pi_p(\lambda), 0]$. Finally, $G(\lambda, x) : \mathbb{R} \rightarrow \mathbb{R}$ is defined as the $2\pi_p(\lambda)$ -periodic extension of $\tilde{G}(\lambda, x)$ to all \mathbb{R} . It follows from this construction that, $G(\lambda, x) = 0 \Leftrightarrow x = n\pi_p(\lambda)$, $n = 0, \pm 1, \pm 2, \dots$. Moreover, we obtain from (3.5) that $u(x) = cG(\lambda, x)$ verifies the equation $-((u')^{p-1})' = \lambda u^{p-1}$ and the initial condition $u(0) = 0$. Finally, to get a solution of (3.3) we use the condition $u(1) = 0$. Now, $u(1) = 0 \Rightarrow G(\lambda, 1) = 0 \Rightarrow n\pi_p(\lambda) = 1 \Rightarrow \pi_p(\lambda) = \frac{1}{n}$. Clearly, by using the substitution $s = \frac{r}{\lambda^{1/p}}$ we obtain $\pi_p(\lambda) = \frac{\pi_p(1)}{\lambda^{1/p}}$. So $\pi_p(\lambda) = \frac{1}{n}$ implies $\lambda_n = (n\pi_p(1))^{1/p}$. Since this argument will not work for problem (3.1), we modify this in the following way: we avoid constructing an exact solution for the equation $\pi_p(\lambda) = \frac{1}{n}$ and instead we show that the eigenvalues of problem (3.3) consist of a sequence λ_n and $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$. Indeed, we have $\lim_{\lambda \rightarrow 0^+} \pi_p(\lambda) = +\infty$, $\lim_{\lambda \rightarrow +\infty} \pi_p(\lambda) = 0$ and $\pi_p(\lambda)$ is a decreasing function on $(0, +\infty)$. It follows from these properties that, for each $n \in \mathbb{N}$ the equation $\pi_p(\lambda) = \frac{1}{n}$ has a unique solution λ_n and $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$.

Now, we demonstrate these modified techniques on the following model problem to get analytical solutions and dispersion relations.

3.1 A Model Problem: $p = 4$ and $k \neq 0$.

In this Subsection we set $p = 4$ in (3.1) and try to find its analytical solution and some dispersion relations between k and λ for the following equation:

$$\begin{aligned} k^2(k^2u^2 + u'^2)u - ((k^2u^2 + u'^2)u')' &= \lambda u^3 \\ u(0) &= u(1) = 0, \end{aligned} \tag{3.6}$$

where $u = u(x)$. In what follows we also assume $k \geq 0$ because of the symmetry property with respect to k .

First, we reduce the order of the equation by the substitution $u' = v(u)$. Then, $u'' = v' = \frac{dv}{du} \frac{du}{dx} = \frac{dv}{du}v$ and we have

$$(3v^3 + k^2u^2v) \frac{dv}{du} = k^4u^3 - k^2uv^2 - \lambda u^3.$$

This is a homogeneous ordinary differential equation and it may be integrated by changing from $(u, v) \rightarrow (u, w)$ by the standard substitution $v = uw(u)$. This then gives a separable equation

$$(3w^3 + k^2w)uw' = k^4 - \lambda - 2k^2w^2 - 3w^4,$$

which integrates to

$$k^4 - \lambda - 2k^2w^2 - 3w^4 = \frac{c}{u^4}.$$

Putting the inverse substitution $w = \frac{u'}{u}$ into the above equation we obtain

$$k^4 - \lambda - 2k^2\left(\frac{u'}{u}\right)^2 - 3\left(\frac{u'}{u}\right)^4 = \frac{c}{u^4}$$

or

$$3u'^4 + 2k^2u^2u'^2 - (k^4 - \lambda)u^4 - c = 0$$

through the multiplication by $-u^4$. Finally, by solving this quadratic equation in u'^2 for u' we get

$$u' = \left[-\frac{k^2}{3}u^2 + \sqrt{\frac{k^4}{9}u^4 + \frac{1}{3}((k^4 - \lambda)u^4 + c)} \right]^{1/2}.$$

We notice that if $k = 0$ and $p = 4$ then this equation and equation (3.4) coincide. Hence, integrating the last equation and using the initial condition $u(0) = 0$ gives

$$\int_0^{u(x)} \frac{dr}{\left[-\frac{k^2}{3}r^2 + \sqrt{\frac{k^4}{9}r^4 + \frac{k^4 - \lambda}{3}r^4 + C} \right]^{1/2}} = x.$$

Let $r = C^{1/4}s$. Then we have

$$\int_0^{cu(x)} \frac{ds}{\left[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda - k^4}{3}s^4} \right]^{1/2}} = x. \quad (3.7)$$

Again, in the case of $k = 0$ we obtain from this equation that

$$\int_0^{cu(x)} \frac{ds}{\left[1 - \frac{\lambda}{3}s^4 \right]^{1/4}} = x,$$

which is the same as (3.5). Let us define

$$F_{\lambda,k}(x) := \int_0^x \frac{ds}{\left[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda - k^4}{3}s^4} \right]^{1/2}}. \quad (3.8)$$

By Theorem 2.1 we have $|k|^p \leq \lambda_1(k)$. Hence, $k^4 \leq \lambda_1(k)$. For this reason in what follows we suppose that $k^4 < \lambda$ (the case $\lambda = k^4$ is considered in Theorem 3.1). Function $F_{\lambda,k}(x)$ is well defined if $1 - \frac{\lambda-k^4}{3}s^4 > 0$. Therefore, $s < \left(\frac{3}{\lambda-k^4}\right)^{1/4}$ and

$$F_{\lambda,k} : \left[0, \left(\frac{3}{\lambda-k^4}\right)^{1/4}\right] \rightarrow \left[0, \frac{\pi_4(\lambda,k)}{2}\right], \quad k^4 < \lambda,$$

where

$$\pi_4(\lambda, k) = 2 \int_0^{\left(\frac{3}{\lambda-k^4}\right)^{1/4}} \frac{ds}{\left[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda-k^4}{3}s^4}\right]^{1/2}}. \quad (3.9)$$

Since $F'_{\lambda,k}(x) > 0$, then there exists an inverse function $G_{\lambda,k} : \left[0, \frac{\pi_4(\lambda,k)}{2}\right] \rightarrow \left[0, \left(\frac{3}{\lambda-k^4}\right)^{1/4}\right]$ defined by

$$\int_0^{G_{\lambda,k}(x)} \frac{ds}{\left[-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda-k^4}{3}s^4}\right]^{1/2}} = x. \quad (3.10)$$

Finally $G_{\lambda,k}(x)$ is extended to \mathbb{R} in the same way as applied to $G(\lambda, x)$ above. By this extension we have $G_{\lambda,k}(x) = 0 \Leftrightarrow x = n\pi_4(\lambda, k)$. Now, it follows from (3.8) and (3.11) that $u(x) = cG_{\lambda,k}(x)$ is the general solution for the following equation

$$k^2(k^2u^2 + u'^2)u - ((k^2u^2 + u'^2)u')' = \lambda u^3, \quad u(0) = 0. \quad (3.11)$$

Thus, the main question is: for what values of k and λ are there non-trivial solutions, among $u(x) = cG_{\lambda,k}(x)$, satisfying the condition $u(1) = 0$? The answer to this question has almost been given in the above discussion. We summarize these in the following theorem.

Theorem 3.1 *Let $p = 4$ and $k \neq 0$. In this case we have:*

- a) All eigen-parameters (λ, k) for problem (3.6) lie in the parabola $k^4 < \lambda$;*
- b) For every $k \in \mathbb{R}$, the set of all eigenvalues λ of problem (3.6) consists of a sequence of positive numbers $\lambda_n(k)$ such that $\lambda_n(k) \rightarrow \infty$;*
- c) There is a number $\lambda^* > 0$ such that for each λ satisfying $\lambda^* \leq \lambda$, the set of all eigen-parameters k ($k \geq 0$) of problem (3.6) consists of a finite number of eigen-parameters $k_1(\lambda), k_2(\lambda), \dots, k_{n(\lambda)}(\lambda)$ which belong to the interval $[0, \lambda^{1/4}]$. Moreover, $n(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$;*
- d) In the case of $\lambda < \lambda^*$ problem (3.6) has only trivial solution.*

Proof. a) Let $\lambda \leq k^4$. Then we have $-\frac{k^2}{3}s^2 + \sqrt{\frac{k^4}{9}s^4 + 1 - \frac{\lambda - k^4}{3}s^4} > 0$ for all $s \in \mathbb{R}$. Hence, it follows from (3.10) that the function $F_{\lambda,k}(x)$ is well defined, positive function on $[0, +\infty)$ and by definition so is the function $G_{\lambda,k}(x)$. Since the solution of (3.11) is given by $u(x) = cG_{\lambda,k}(x)$, then $u(1) = 0$ implies $c = 0$. Thus, all eigen-parameters (λ, k) for problem (3.6) lie in the parabola $k^4 < \lambda$ (see [13] for more facts on the localization of the eigen-parameters).

b) Actually, this part has already been proved in Theorem 2.1. In our case we just present an alternative method for the proof. By (3.9), for a fixed k the function $\pi_4(\lambda, k)$ satisfies the following conditions: $\lim_{\lambda \rightarrow k^4+} \pi_4(\lambda, k) = +\infty$, $\lim_{\lambda \rightarrow +\infty} \pi_4(\lambda, k) = 0$ and $\pi_4(\lambda, k)$ is a decreasing function on $(k^4, +\infty)$. It follows from these properties that, for each $n \in \mathbb{N}$ the equation $\pi_4(\lambda, k) = \frac{1}{n}$ has a unique solution λ_n and $\lambda_n \rightarrow +\infty$ as $n \rightarrow \infty$.

c) Now, let us fix $\lambda > 0$. Then $\pi_4(\lambda, \cdot)$ is defined on the interval $[0, \lambda^{1/4}]$. Moreover,

$$\pi_4(\lambda, 0) = 2 \int_0^{(\frac{3}{\lambda})^{1/4}} \frac{ds}{(1 - \frac{\lambda}{3}s^4)^{1/4}},$$

and $\lim_{k \rightarrow \lambda^{1/4}-} \pi_4(\lambda, k) = +\infty$. Let λ^* be the solution of the equation $\pi_4(\lambda, 0) = 1$. $\pi_p(\lambda, 0)$ is a decreasing function and $\pi_p(\lambda, 0) \rightarrow 0$ as $\lambda \rightarrow +\infty$. Now, clearly the equation $\pi_4(\lambda, k) = \frac{1}{n}$ has a solution if and only if $\lambda^* \leq \lambda$ and $\frac{1}{n} \in [\pi_4(\lambda), 1]$. These facts prove c) and d).

3.2 On the analytical solutions for arbitrary $p > 1$

In the previous subsection we have constructed analytical solutions to (3.1) for some special cases. Unfortunately, this method can not be applied to problem (3.1) for arbitrary $p > 1$. Indeed if we try to repeat the same steps from the case $p = 4$ then by using the substitution $v = v(u)$ we can reduce the order of the equation

$$k^2 |\nabla_k u|^{p-2} u - (|\nabla_k u|^{p-2} u')' = \lambda |u|^{p-2} u$$

and by solving the reduced equation as a homogeneous, 1st order ordinary differential equation we get

$$u = \frac{(k^2 + \frac{u'^2}{u^2})^{1/p} \left(-k^2 (k^2 + \frac{u'^2}{u^2})^{p/2} - \frac{u'^2 (k^2 + \frac{u'^2}{u^2})^{p/2}}{u^2} + \frac{p u'^2 (k^2 + \frac{u'^2}{u^2})^{p/2}}{u^2} + \lambda (k^2 + \frac{u'^2}{u^2}) \right)^{-\frac{1}{p}}}{c}.$$

However, this equation is not a radically solvable algebraic equation with respect to u' . Therefore, in this case we have to use a different method. We study these questions in the next section.

4 On the dispersion relation $k(\lambda)$ for arbitrary $p > 1$: A variational approach

This Section is devoted to the dispersion relation $k(\lambda)$, when the problem is not analytically solvable. In this case we do not need to restrict ourselves to one dimensional problems. Our problem is

$$k^2 \int_{\Omega} |\nabla_k u|^{p-2} uv \, dx + \int_{\Omega} |\nabla_k u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} |u|^{p-2} uv \, dx, \quad p > 1, \quad (4.1)$$

where Ω is a bounded domain in \mathbb{R}^n , $u \in W_0^{1,p}(\Omega)$, $\nabla_k u := (ku, \nabla u)$ and $\nabla u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$. The approach we use below is based on the methods of operator pencils. Now we briefly describe the method of operator pencils which has widely been used in the spectral theory of linear operator pencils (see [15] and [6]). An operator pencil is an operator-valued function and particularly, it is a polynomial with coefficients in a space of linear operators. Typical eigenvalue problems for operator pencils are:

$$k^n A_n u + k^{n-1} A_{n-1} u + \dots + k A_1 u + A_0 u = 0$$

or

$$k^n A_n u + k^{n-1} A_{n-1} u + \dots + k A_1 u + A_0 u = \lambda D u.$$

In the case $k = 2$ we deal with a quadratic eigenvalue problem and we are going to present variational techniques in this case. Thus we have a quadratic eigenvalue problem

$$k^2 A u + k B u + C u = \lambda D u. \quad (4.2)$$

Let us consider the equation

$$k^2 (A u, u) + k (B u, u) + (C u, u) = \lambda (D u, u).$$

This equation defines the following functionals

$$r_{\pm}(u, \lambda) = \frac{-(B u, u) \pm \sqrt{(B u, u)^2 - 4((C - \lambda D) u, u)(A u, u)}}{2(A u, u)}$$

which play a central role in the variational theory of the eigenvalue problems (4.2). Actually, in this theory the functionals $r_{\pm}(u, \lambda)$ are play the same role as the functionals $\Phi(u)$ and $G_k(u)$ (see Section 2) in the Ljusternic-Schnirelman critical point theory. It turns out that all variational characterizations for k for problem (4.2) are obtained via $r_{\pm}(u, \lambda)$. Namely, under some additional conditions we have (see [6] and references therein):

$$k_n^\pm(\lambda) = \inf_{\substack{L \subset X \\ \dim L = n}} \sup_{\substack{u \in L \\ x \neq 0}} r_\pm(u, \lambda). \quad (4.3)$$

Now we follow this method for nonlinear problems. We have seen a simple connection between problems (4.1) and (4.2) in the case $p = 4$. We have established a connection between k and λ by the following differential equations (see the equation next to (3.6)):

$$k^4 u^3 - (uu'^2 + u^2 u'')k^2 - u'^2 u'' = \lambda u^3,$$

which can be written in the operator pencil form (we set $k^2 := \nu$)

$$\nu^2 Au + \nu Bu + Cu = \lambda Au, \quad (4.4)$$

where $Au = u^3$, $Bu = -(uu'^2 + u^2 u'')$ and $Cu = -u'^2 u''$.

Although the operators A , B and C in (4.4) are non-linear we can extend many methods applied in the spectral theory of the operator pencils to the non-linear eigenvalue problems, including problem (4.1) and its particular case (4.4). Below we give some results in this direction.

If we replace k^2 by ν in (4.1) then

$$\begin{aligned} & \nu \int_{\Omega} (\nu u^2 + |\nabla u|^2)^{\frac{p-2}{2}} uv \, dx + \int_{\Omega} (\nu u^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \cdot \nabla v \, dx \\ & = \lambda \int_{\Omega} |u|^{p-2} uv \, dx, \end{aligned} \quad (4.5)$$

where $\nu \geq 0$. Equation (4.5) is the variational equation for the functional $F_{\lambda, \nu}(u) = \int_{\Omega} (\nu u^2 + |\nabla u|^2)^{\frac{p}{2}} \, dx - \lambda \int_{\Omega} |u|^p \, dx$. Let us fix $\lambda > 0$ and define $f_{\lambda}(\nu, u) := F_{\lambda, \nu}(u)$. Evidently, $f_{\lambda} : \mathbb{R} \times X \rightarrow \mathbb{R}$ is a continuously differentiable functional. The equation $f_{\lambda}(\nu, u) = 0$ (for a fixed λ) defines the so-called root functional $r_{\lambda}(u) := \nu$ with values in an interval $J \subset [0, \sqrt[p]{\lambda^2}]$, which plays the same role as $r_{\pm}(u, \lambda)$ for (4.2). We note that the equation $f_{\lambda}(\nu, u) = 0$ in general defines several functionals and each functional describes the eigenvalues which belong to its range.

Next, we give the basic relation between the problem (4.5) and the root functional $r_{\lambda}(u)$ (see also [13]). In what follows we fix λ and by eigenvalues we mean a parameter ν (or the same k^2), satisfying (4.5) with a non-trivial $u \in W_0^{1,p}(\Omega)$.

Theorem 4.1 *a) $r(u) \in C^1(X \setminus \{0\}, J)$ and it is extended as a continuous mapping on X by setting $r(0) = 0$,*

- b) (λ, ν) , $\nu \in J$ is an eigen-pair, corresponding to the eigenvector u for problem (4.5) if and only if u is a critical point and ν is a critical level for r , i.e., $\langle r'(u), v \rangle = 0$ for all $v \in X$ and $r(u) = \nu$,
c) All eigenvalues lie in the parabola $\nu^{\frac{p}{2}} < \lambda$,
d) If J is closed, then the end points of the interval J are eigenvalues of problem (4.5).

Proof. a) As

$$\frac{d}{d\nu} f_\lambda(\nu, u) \Big|_{\nu=r(u)} = \frac{p}{2} \int_{\Omega} (r(u)u^2 + |\nabla u|^2)^{\frac{p-2}{2}} u^2 dx > 0, \quad 0 \neq u \in X,$$

it follows from the implicit function theorem that $r \in C^1(X \setminus \{0\}, J)$ (see [19], vol I, p.149).

b) By the definition of $r(u)$, we have

$$\int_{\Omega} (r(u)u^2 + |\nabla u|^2)^{\frac{p}{2}} dx = \lambda \int_{\Omega} |u|^p dx.$$

Taking the Fréchet derivative from both sides, we obtain

$$\begin{aligned} & \frac{p}{2} \int_{\Omega} (r(u)u^2 + |\nabla u|^2)^{\frac{p-2}{2}} \left[\langle r'(u), v \rangle u^2 + 2r(u)uv + 2\nabla u \cdot \nabla v \right] dx \\ &= \lambda \int_{\Omega} |u|^{p-2} uv dx. \end{aligned}$$

By regrouping the terms, we get

$$\begin{aligned} & \frac{p}{2} \langle r'(u), v \rangle \int_{\Omega} |\nabla_{r(u)} u|^{p-2} u^2 dx + p \left[r(u) \int_{\Omega} |\nabla_{r(u)} u|^{p-2} uv dx + \right. \\ & \left. \int_{\Omega} |\nabla_{r(u)} u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{p-2} uv dx \right] = 0, \end{aligned}$$

or

$$\frac{1}{2} \langle r'(u), v \rangle \int_{\Omega} |\nabla_{r(u)} u|^{p-2} u^2 dx + \langle F'_{\lambda, r(u)}(u), v \rangle = 0.$$

Finally,

$$\langle r'(u), v \rangle = \frac{-2 \langle F'_{\lambda, r(u)}(u), v \rangle}{\int_{\Omega} |\nabla_{r(u)} u|^{p-2} u^2 dx}, \quad u \neq 0. \quad (4.6)$$

and by the definition of $F'_{\lambda, \nu}(u)$ a pair of numbers (λ, ν) is an eigen-pair if and only if $\langle F'_{\lambda, \nu}(u), v \rangle = 0$. The needed results follow from (4.6).

c) This fact immediately follows from the inequality

$$\nu^{\frac{p}{2}} \int_{\Omega} |u|^p dx < \int_{\Omega} (\nu u^2 + |\nabla u|^2)^{\frac{p}{2}} = \lambda \int_{\Omega} |u|^p dx,$$

where $u \neq 0$.

d) The closeness of J means that $\inf r(u)$ and $\sup r(u)$ attains. Consequently, these points are critical levels for the functional $r(u)$. \square

Finally, there are a finite number of eigenvalues for problem (4.1) denoted by $k_1(\lambda), k_2(\lambda), \dots, k_{n(\lambda)}(\lambda)$ (see Theorem 3.1), which are described by

$$k_n(\lambda) = \inf_{K \subset \mathcal{K}_n} \sup_{u \in K} r_\lambda(u).$$

For this it is enough to check the Palais-Smale condition for $r_\lambda(u)$ in the interval $J \subset [0, \sqrt[p]{\lambda}]$ or conditions $H1 - H4$ given in Section 2.

References

- [1] R. A. Adams, J. J. F. Fournier, Sobolev spaces, Academic Press, 2002.
- [2] A. Anane, Simplicité et isolation de la première valeur du p -Laplacian avec poids, *C.R. Acad. Sci. Paris Ser. I Math.* **305**, (1987), 725-728.
- [3] A. Anane, N. Tsouli, On the second eigenvalue of the p -Laplacian, *Nonlinear Partial Differential Equations*, Pitman Research Notes Mathematics Series vol. **343**, Longman, Harlow, (1996), pp.1-9.
- [4] J.P. G. Azorero, I. P. Alonso, Existence and nonuniqueness for the p -Laplacians: Nonlinear eigenvalues, *Commun. in Partial Differential Equations*, **12**, (1987), 1389- 1430.
- [5] J. Benedikt, P. G. Girg, P. Takác, On the Fredholm alternative for the p -Laplacian at higher eigenvalues (in one dimension), *Nonlinear Analysis, Theory Methods and Applications*, **72** (2010), 3091-3107.
- [6] N. Colakoglu, M. Hasanov, B. U. Uzun, Eigenvalues of two parameter polynomial operator pencils of waveguide type, *Integral Equations Operator Theory*, **56**, (2006), 381- 400.
- [7] E. Di Benedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Analysis*, **7**, (1983), 827-850.
- [8] E. Di Benedetto, M.A. Herrero, Non-negative solutions of the evolution p -Laplacian equation *Archive for Rational Mechanics and Analysis*, **111**, No. 3 (1990), 225-290.

- [9] An Lê, Eigenvalue problems for the p -Laplacians, *Nonlinear Analysis, Theory methods and Applications* **64**, (2006), 1057-1099.
- [10] P. Drábek, Ranges of α -homogeneous operators and their perturbations, *Casapis Pest. Mat.* **105** (1980), 167-183.
- [11] P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, de Gruyter series in Nonlinear Analysis and applications, vol.5. Walter de Gruyter co., Berlin, 1997
- [12] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential equations of Second Order, Classics in Mathematics*, Springer, Berlin, 2001.
- [13] M. Hasanov, Eigenvalue problems for perturbed p -Laplacians, *American Institute of Physics, Conference Proceedings series*, vol.**1309**, pp.400-411, 2010.
- [14] P. Lindqvist, On the equation $-\operatorname{div}(|\nabla u|^{p-2}|\nabla u|) = \lambda|u|^{p-2}$, *Proceedings of AMS*, **109**, No. 1, (1990), 157- 164.
- [15] A. S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Translations of Mathematical Monographs, vol.**71**, American Mathematical Society, Providence, RI, 1988.
- [16] M. Del Pino, M. Elgueta, R. Manasevich, A homotopic Deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0$, $u(0) = u(T) = 0$, $p > 1$, *Journal of Differential Equations*, **80**, (1989), 1-13.
- [17] M. Struwe, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Fourth Edition, Springer, 2002.
- [18] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *Journal of Differential Equations*, **51** (1984) 126-150.
- [19] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. I- Fixed point theory, Vol. III., Variational Methods and Optimization*, Springer-Verlag, 1985.